Start with the substitution $z = \sin x$ and then integrate by parts …

$$I = \int_0^1 \frac{\sin^{-1} z}{z}\, dz = \int_0^{\pi/2} \frac{x}{\sin x} \cos x \, dx$$
$$= \int_0^{\pi/2} x \frac{d}{dx}(\ln(\sin x))\, dx$$
$$= \Big[x \ln(\sin x)\Big]_0^{\pi/2} - \int_0^{\pi/2} \ln(\sin x)\, dx$$
$$= -\int_0^{\pi/2} \ln(\sin x)\, dx.$$

There is a cute trick for this integral. Since $\int_0^{\pi/2} \ln(\sin x)\, dx = \int_{\pi/2}^{\pi} \ln(\sin x)\, dx$,

$$I = -\frac{1}{2}\int_0^{\pi} \ln(\sin x)\, dx.$$

Now make the substitution $x = 2y$

$$I = -\int_0^{\pi/2} \ln\left(\sin 2y\right) dy$$
$$= -\int_0^{\pi/2} \ln(2 \sin y \cos y)\, dy$$
$$= -\int_0^{\pi/2} \ln 2\, dy - \int_0^{\pi/2} \ln(\sin y)\, dy - \int_0^{\pi/2} \ln(\cos y)\, dy$$
$$= -\frac{\pi}{2}\ln 2 + 2I$$

since clearly $\int_0^{\pi/2} \ln(\sin y)\, dy = \int_0^{\pi/2} \ln(\cos y)\, dy$. So $I = \frac{\pi}{2}\ln 2$.