Integral of the Week 3

Solution by Simon Tyler

Find the integral

$$I = \int W(1/z)\mathrm{d}z \ ,$$

where $W(z)$ is the Lambert-W function defined by

$$z = W(z)\mathrm{e}^{W(z)} \quad \Longrightarrow \quad W(0) = 0 \quad \text{and} \quad zW'(z) = \frac{W(z)}{1+W(z)} \ .$$

A couple of its other cute properties are that

$$\frac{\mathrm{d}}{\log z}\log W(z) = \frac{1}{1+W(z)} \quad \text{and} \quad \log W(z) = \log z - W(z) \ ,$$

but it turns out we don't need any of these!

Note that below all integrals are up to a constant, and it's assumed that $W(z)$ is always on its principle branch. First we rewrite I by using $x = 1/z$

$$I = -\int \frac{1}{x^2} W(x)\mathrm{d}x \ .$$

Then substitute

$$w = W(x) \iff x = w\mathrm{e}^w \quad \Longrightarrow \quad \mathrm{d}x = (1+w)\mathrm{e}^w \mathrm{d}w$$

to get

$$I = -\int \frac{\mathrm{e}^{-w}}{w}(1+w)\mathrm{d}w = \mathrm{e}^{-w} + E_1(w) \ ,$$

where $E_n(z)$ is the exponential integral[1]

$$E_n(z) \stackrel{\mathrm{Re}(z)>0}{=\!=\!=} \int_1^\infty \frac{1}{s^n}\mathrm{e}^{-zs}\mathrm{d}s \ , \quad \frac{\mathrm{d}}{\mathrm{d}z}E_n(z) = -E_{n-1}(z) \ , \quad zE_0(z) = \mathrm{e}^{-z} \ .$$

We now simply sub back in $w = W(x) = W(1/z)$ and use the defining relation $\mathrm{e}^{-W(1/z)} = zW(1/z)$ to get the result (that is easily checked by differentiation)

$$I = \int W(1/z)\mathrm{d}z = zW(1/z) + E_1(W(1/z)) \ .$$

We can generalise the whole procedure to

$$\begin{aligned} J_{a,b}(z) &= \int z^{a-1} W\left(\frac{1}{z}\right)^b \mathrm{d}z = -\int \frac{1}{x^{a+1}} W(x)^b \mathrm{d}x \\ &= -\int w^{b-a-1}(1+w)\mathrm{e}^{-aw}\mathrm{d}w = \frac{-b}{b-a}\int w^{b-a}\mathrm{e}^{-aw}\mathrm{d}w - \frac{w^{b-a}}{b-a}\mathrm{e}^{-aw} \\ &= \frac{w^b}{b-a}\left(bw^{1-a}E_{a-b}(aw) - \frac{1}{x^a}\right) \ , \quad w = W(x) = W(1/z) \ . \end{aligned}$$

Due to the integration by parts the limit $I = \lim_{a\to 1} J_{a,1}(x)$ has to be taken carefully, but it does work. Maybe a better form is without the IBP to get

$$J_{a,b}(z) = w^{b-a}\left(E_{1+a-b}(aw) + wE_{a-b}(aw)\right) \ , \quad w = W(1/z) \ .$$

[1] It is related to the incomplete gamma function via $E_n(z) = z^{a-1}\Gamma(1-z, z)$.